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Control under communication constraints

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Abstract

We consider variations of the simplest one-dimensional linear control system in which communication between the sensor and the controller is constrained by a binary communication channel. The link between the controller and the plant is not constrained. We study the limits imposed by the channel on the controller's ability to estimate the state and achieve stability. For discrete-time systems, several such limits have been published in the literature. We first derive these limits as bounds on the state transition matrix of the linear system then we investigate extensions for the cases (1) unknown initial conditions, (2) unknown state transition matrix and (3) noisy binary communication channels. For continuous-time systems, we consider a binary queue as the communication link. For this case, we also derive the bounds on the gain matrix of the linear system. To our knowledge this is the first time a binary queue has been studied as the communication link in the proposed setting.

1 Introduction

We consider variations of the simplest one-dimensional discrete-time linear control system in which communication between the sensor and the controller is constrained by a binary communication channel. The link between the controller and the plant is not constrained. We study the limits imposed by the channel on the controller's ability to estimate the state and achieve stability.

We first reproduce the results of Tatikonda for discrete-time systems and demonstrate that $|a| < 2$ (where a is the transition matrix of the linear system) is a necessary and sufficient condition for achieving stability for a noiseless binary channel (Theorems 1,2). This result is valid when a is known to the controller. We continue the investigation by looking at noisy binary channels and derive similar bounds on $|a|$ (Theorem 3). We then look at the case where a is not known to the controller and propose a different bound on the plant (Theorem 4). We conclude our investigation of discrete-time systems by demonstrating the relationship between the Shannon capacity of an arbitrary channel and different modes of stability.

We next study a continuous-time version of the system with the same technical setup as above: the link between the sensors and the controller is a binary communication channel. In the spirit of Astrom [4] we consider an *event*-based control policy to compare the fundamental bounds imposed on the plant and the controller. Astrom compares such an event-based continuous-time controller with the discretized LQG controller and demonstrates that the *system variance-to-control intervention frequency ratio* is on average better for the event-based controller. However, the comparison excludes several information-theoretic aspects of both the discretized system and the continuous event-based system.

We probe further into this example by taking the communication link to be a queue. In particular we produce a stability result that is completely characterized by the service distribution of the queue (Theorem 5) and we discuss the information-theoretic implications of this result. Since queueing networks are good mathematical models of many communication networks, our setup offers an alternate approach to the study of *networked control systems*, which was the subject of a recent issue of the IEEE Control Systems Magazine [5].

2 Discrete-time systems

We start with a first-order, unstable discrete-time system,

$$x_{k+1} = ax_k + w_k, \; k \geq 0, \tag{1}$$

where $w_k \in [-W, W]$ is a bounded unknown disturbance. An *encoder* emits a binary signal $z_k \in \{-1, 1\}$ based on the observations $x^k = (x_0, \cdots, x_k)$ up to time k. So the encoder is described by functions ϕ_k,

$$z_k = \phi_k(x^k), \; k \geq 0.$$

At time k the *decoder* receives z_k. Based on its observations $z^k = (z_0, \cdots, z_k)$, the decoder makes an estimate $\hat{x}_k$ of x_k. So the decoder is described by functions ψ_k,

$$\hat{x}_k = \psi_k(z^k), \; k \geq 0.$$

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The problem is to design $\{\phi_k, \psi_k\}$ so that the error in the estimate is bounded, or to determine if no such encoder-decoder pair exists.

Theorem 1 (Tatikonda) Suppose x_0 is unknown but bounded, $|x_0| \leq \Delta$. Suppose the decoder knows a. If $|a| < 2$, there is a bounded-error encoder-decoder pair so that

$$\sup_{k\geq 0} |x_k - \hat{x}_k| < \infty,$$

If $|a| > 2$ no such pair exists.

Proof We construct the required pair. At each k, the decoder determines an uncertainty interval $[l_k, u_k]$ that is *guaranteed* to contain x_k and selects the estimate to be the mid-point of this interval, $\hat{x}_k = \frac{1}{2}(l_k + u_k)$, starting with $[l_0, u_0] = [-\Delta, \Delta]$. Observe that at each time k, the encoder can also calculate $\hat{x}_k$. The encoder selects z_{k+1} as $z_{k+1} = 1$ if $x_{k+1} \geq a\hat{x}_k$ and $z_{k+1} = -1$, otherwise. Upon receiving z_{k+1}, the decoder selects $[l_{k+1}, u_{k+1}]$ by

$$[l_{k+1}, u_{k+1}] = \begin{cases} [a\hat{x}_k, au_k + W] & \text{if } z_{k+1} = 1 \\ [al_k - W, a\hat{x}_k] & \text{if } z_{k+1} = -1 \end{cases} \tag{2}$$

Since $x_{k+1} = ax_k + w_k \in a[l_k, u_k] + [-W, W]$ it follows that $x_{k+1} \in [l_{k+1}, u_{k+1}]$. The size of the uncertainty interval evolves as

$$u_{k+1} - l_{k+1} = \frac{a}{2}(u_k - l_k) + W,$$

which remains bounded if $|a| < 2$.

On the other hand, at time k, $x_k = \sum_{j=0}^{k-1} a^{k-1-i} w_j$ can be any point in the interval $X_k = \frac{|a|^k - 1}{|a| - 1}[-W, W]$. By time k the decoder has observed 2^k possible values of z^k, based on which it could have made 2^k possible point estimates $\psi_k(z^k)$. Hence the worst estimation error is at least

$$\max_{x\in X_k} \min_{\psi_k} |x - \psi_k(z^k)| \geq 2W \frac{|a|^k - 1}{|a| - 1} \frac{1}{2^{k+1}},$$

which is unbounded if $|a| > 2$. □

Now suppose that x_0 is completely unknown. Tatikonda [1] proposes a two-phase version of the encoder-decoder pair to handle this case. In the first phase, the state space is *explored* to find an interval $[l_{\hat{k}}, u_{\hat{k}}]$ that is *guaranteed* to contain $x_{\hat{k}}$. In the second phase the encoder-decoder pair is applied with the starting time $k = \hat{k}$ instead of $k = 0$. However, his method requires the use of an auxiliary *marker* symbol to indicate the end of the exploration phase, i.e. the time $\hat{k}$.

Corollary 1 Suppose x_0 is completely unknown. Suppose $|a| < 2$, and suppose the decoder knows a. Then there is an asymptotically bounded-error encoder-decoder pair) so that

$$\limsup_{k\to\infty} |x_k - \hat{x}_k| < \infty.$$

If $|a| > 2$ no such pair exists.

Proof If $|a| > 2$ Theorem 1 implies that no such pair exists. Take $|a| < 2$. Once again, at each k, the decoder determines an uncertainty interval $[l_k, u_k]$ that is *eventually* guaranteed to contain x_k and selects the estimate to be the mid-point of this interval, $\hat{x}_k = \frac{1}{2}(l_k + u_k)$, starting with $[l_0, u_0] = [-1, 1]$.

If k is a power of 2 (i.e. for $k = 1, 2, 4, 8, \ldots$) the encoder selects $z_{k+1} = 1$ if $x_k \in [l_k, u_k]$ and $z_{k+1} = -1$ otherwise. If k is not a power of 2 the encoder selects $z_{k+1} = 1$ if $x_{k+1} \geq a\hat{x}_k$ and $z_{k+1} = -1$, otherwise.

If k is a power of 2, upon receiving z_{k+1}, the decoder selects $[l_{k+1}, u_{k+1}]$ by

$$[l_{k+1}, u_{k+1}] = \left[-2^{k+1} - \sum_{i=0}^{k} 2^i W, 2^{k+1} + \sum_{i=0}^{k} 2^i W\right]$$

if $z_{k+1} = 1$ and

$$[l_{k+1}, u_{k+1}] = [al_k - W, au_k + W]$$

if $z_{k+1} = -1$. When k is not a power of 2, the decoder selects $[l_{k+1}, u_{k+1}]$ as in (2). Since $|a| < 2$, there is some power of 2, denote it by $\hat{k}$, so that the interval

$$\left[-\left(2^{\hat{k}+1} - \sum_{i=0}^{\hat{k}} 2^{\hat{k}-i} W\right), 2^{\hat{k}+1} + \sum_{i=0}^{\hat{k}} 2^{\hat{k}-i} W\right]$$

contains $x_{\hat{k}+1}$. Furthermore, for all $k > \hat{k}$ it follows that $x_k \in [l_k, u_k]$. For any power of 2 greater than $\hat{k}$, the encoder will always pick $z_{k+1} = -1$, so for $k > \hat{k}$ the size of the uncertainty interval evolves as

$$u_{k+1} - l_{k+1} = \begin{cases} a(u_k - l_k) + 2W & \text{if } k \text{ a power of 2} \\ \frac{a}{2}(u_k - l_k) + W & \text{otherwise} \end{cases}$$

from which it follows that $|x_k - \hat{x}_k|$ will be asymptotically bounded when $|a| < 2$. □

We have considered a state estimator for the system (1) that is restricted by receiving information over a *digital* communication channel with capacity 1 bps. We have derived the bound $|a| < 2$ that is both sufficient and necessary for the estimation error to remain bounded. We next show that, due to the linearity of the system, controlling the system in closed loop over the digital channel has no effect on these bounds. We then proceed to derive the bounds imposed on the system for the *binary erasure* and *binary symmetric* channel which, unlike the digital channel, have a non-zero probability of error, $p(\hat{z_k} \neq z_k) > 0$. From Theorem 1 and Corollary 1 we see that the bounds on $|a|$ are not changed whether $|x_0| < \Delta$ or x_0 is completely unknown, so for simplicity, we will no longer consider the latter.

2.1 Stabilizing known plant over digital channel
Instead of (1) consider the controlled system

$$x_{k+1} = ax_k + u_k + w_k, \tag{3}$$

where u_k is selected by the decoder, based on the observations z^k. If $|a| < 2$ there is an encoder-decoder pair with bounded estimation error. Because the system is linear, any feedback policy $u_k = Fx_k$ based on the true state which keeps the state bounded, will also keep the state bounded when applied to the estimate $u_k = F\hat{x}_k$. On the other hand, if $|a| > 2$ no controller can keep the state bounded. For suppose $u_k = g_k(z^k)$, $k \geq 0$, is a stabilizing controller. But then the zero-state response of the system (3),

$$\hat{x}_k = \sum_{i=0}^{k-1} a^{k-1-i} g_i(z^i), \; k \geq 0,$$

is a bounded-error estimator for (1) which contradicts Theorem 1, and proves the following result. (See also Tatikonda where the optimal LQG controller is shown to have the *certainty equivalence* property.)

Theorem 2 The controlled system (3) can be stabilized if $|a| < 2$. If $|a| > 2$, no such controller exists.

2.2 Estimation over the binary erasure and binary symmetric channels
The binary digital channel is a useful theoretical tool. However, it is not a realistic model for most physical systems of interest. For example, a noisy wireless communication link or the internet. The binary erasure (BEC) and binary symmetric channels (BSC) are more realistic models for many systems.

Consider again the controlled system (1) with $w_k \in [-W, W]$ and $|x_0| < \Delta$. The encoder emits a binary signal $z_k \in \{-1, 1\}$ based on the observations $x^k = (x_0, \cdots, x_k)$ up to time k. Fix an error probability $e > 0$. At time k the decoder receives $\hat{z}_k$ where for the BEC

$$p(\hat{z}_k = z_k) = 1 - e, \; p(\hat{z}_k = \emptyset) = e,$$

and for the BSC

$$p(\hat{z}_k = z_k) = 1 - e, \; p(\hat{z}_k = \overline{z_k}) = e,$$

independently for each k. For the BEC, the $\emptyset$ means that no symbol was received by the decoder at time k. So, if a symbol is received by the decoder, then it is received correctly. The BSC is slightly different in that a symbol is always received by the decoder but there is a non-zero probability that the received symbol may be flipped, where $\overline{z_k}$ denotes the complement of z_k. Based on its observations $\hat{z}^k = (\hat{z}_0, \cdots, \hat{z}_k)$, the decoder makes an estimate $\hat{x}_k$ of x_k.

Theorem 3a (Tatikonda) Consider the system (1) and the BEC. Suppose the decoder knows a and the encoder knows when the erasures have occured. If $|a| < 2^{1-e}$ then there exists a bounded-error encoder-decoder pair so that

$$\hat{x}_k \to x_k \text{ with probability 1.}$$

If $|a| > 2^{1-e}$ no such pair exists.

Proof We prove sufficiency. At each k, the decoder determines an uncertainty interval $[l_k, u_k]$ that is guaranteed to contain x_k and selects the estimate to be the mid-point of this interval, $\hat{x}_k = \frac{1}{2}(l_k + u_k)$, starting with $[l_0, u_0] = [-\Delta, \Delta]$. At time k the encoder knows whether an erasure has occured, it can also calculate $\hat{x}_k$. The encoder selects z_{k+1} as $z_{k+1} = 1$ if $x_{k+1} \geq a\hat{x}_k$ and $z_{k+1} = -1$, otherwise. Upon receiving $\hat{z}_{k+1}$, the decoder selects $[l_{k+1}, u_{k+1}]$ by

$$[l_{k+1}, u_{k+1}] = \begin{cases} [a\hat{x}_k, au_k + W] & \text{if } \hat{z}_{k+1} = 1 \\ [al_k - W, a\hat{x}_k] & \text{if } \hat{z}_{k+1} = -1 \\ [al_k - W, au_k + W] & \text{if } \hat{z}_{k+1} = \emptyset \end{cases} \tag{4}$$

Since $x_{k+1} = ax_k + w_k \in a[l_k, u_k] + [-W, W]$ it follows that $x_{k+1} \in [l_{k+1}, u_{k+1}]$. The size of the uncertainty interval evolves as

$$u_{k+1} - l_{k+1} = \begin{cases} a(u_k - l_k) + 2W & \text{if } \hat{z}_k = \emptyset \\ \frac{a}{2}(u_k - l_k) + W & \text{otherwise} \end{cases}$$

Denote by $j(k)$ the number of erasures that have occured by time k. By the law of large numbers, as $k \to \infty$, we have that $j(k) \to (1-e)k$ with probability 1. Thus $\sum_{i=0}^{k} \frac{|a|^i}{2^{j(i)}}$ converges with probability 1 when $|a| < 2^{1-e}$. The result follows. □

Theorem 3b (Sahai) If $|a| < \frac{2}{\sqrt{1+3e}}$ there exists an asymptotically bounded-error encoder-decoder pair so that

$$\lim_{k \to \infty} E(x_k - \hat{x}_k)^2 < \infty. \tag{5}$$

If $|a| > \frac{2}{\sqrt{1+3e}}$ then no such pair exists.

Proof See Sahai [3].

Careful examination of the proof in Sahai [3] reveals the following corollary.

Corollary 2 One may replace the inequality 5 with

$$\lim_{k \to \infty} E(x_k - \hat{x}_k)^m < \infty.$$

provided that

$$|a|^m < \frac{2^m}{1 + (2^m - 1)e}. \tag{6}$$

Note that the Shannon Capacity of the digital channel is $C = 1$ and the BEC is $C = 1 - e$. For almost sure convergence of the estimator we see that for both channels the sufficient and necessary bounds imposed on the system is given by $|a| < 2^C$. On the other hand, there is no visibly clear relation between $|a|$ and C when we require a bound on the m-th moment of the estimation error. It is thus reasonable to conclude that the Shannon capacity of the channel does not provide the right figure of merit for characterizing the desired bounds. Sahai has developed a notion of capacity called the *any-time* capacity for which he produces a visibly simple relation between $|a|$ and C for the convergence of the m-th moment of the estimation error.

The situation is quite different for the BSC. In the proof of Theorem 3a, we used the fact that the decoder can maintain an interval of uncertainty that is *guaranteed* to contain the real state of the system for each k. For the BSC, however, there is no way the decoder can maintain this property. On the other hand, the encoder-decoder pair has to ensure that all errors (or flipped bits) are accounted for, otherwise, the error will magnify by a factor of $|a|^k$ which cannot be bounded for any $|a| > 1$.

We will construct an encoder-decoder pair for which we will show by simulation results to have bounded estimation error. We start with the encoding of 3 symbols $\{-1, 1, B\}$ where B denotes a *back-space*:

symbol	encoding
-1	-1,-1
1	1,1
B	-1,1 or 1,-1

The decoder maintains an uncertainty interval $[l_k, u_k]$ (initially $[l_0, u_0] = [-\Delta, \Delta]$) and estimates the state as $\hat{x_k} = \frac{1}{2}(l_k + u_k)$. The decoder also maintains a backspace count $(\hat{k_0}, N)$ (initially $(\hat{k_0}, N) = (0, 0)$). As we did for the BEC, we will assume that at time k the encoder knows $\hat{z}^k$. Suppose at time k it was the case that $x_k \in [l_k, u_k]$. The encoder marks this time as $k_0 = k$ and selects $(z_{k+1}, z_{k+2}) = (1, 1)$ if $x_{k+1} \geq a^2\hat{x}_k$ and $(z_{k+1}, z_{k+2}) = (-1, -1)$, otherwise. Upon receiving $(\hat{z}_{k+1}, \hat{z}_{k+2})$ the decoder updates its estimate according to

$$[l_{k+2}, u_{k+2}] = \left[a^2\hat{x}_k, a(au_k + W) + W\right] \tag{7}$$

if $(\hat{z}_{k+1}, \hat{z}_{k+2}) = (1, 1)$, and

$$[l_{k+2}, u_{k+2}] = \left[a(al_k - W) - W, a^2\hat{x}_k\right] \tag{8}$$

if $(\hat{z}_{k+1}, \hat{z}_{k+2}) = (-1, -1)$, and (denoting $j = (k+2) - (\hat{k_0} - N)$),

$$[l_{k+2}, u_{k+2}] = \left[a^j l_{\hat{k_0}-N} - \sum_{i=0}^{j-1} a^i W, a^j u_{\hat{k_0}-N} + \sum_{i=0}^{j-1} a^i W\right] \tag{9}$$

otherwise. The backspace count is updated as

$$(\hat{k_0}, N) = \begin{cases} (k, 0) & \text{if } (\hat{z}_{k+1}, \hat{z}_{k+2}) = (1, 1) \\ & \text{or } (\hat{z}_{k+1}, \hat{z}_{k+2}) = (-1, -1) \\ (\hat{k_0}, N+2) & \text{otherwise} \end{cases} \tag{10}$$

If $(\hat{z}_{k+1}, \hat{z}_{k+2}) = (z_{k+1}, z_{k+2})$ then it is easy to see that (see proof of Theorem 1) $x_{k+2} \in [l_{k+2}, u_{k+2}]$. In this case, the encoder will mark $k_0 = k+2$ and repeat the process. If, however, $(\hat{z}_{k+1}, \hat{z}_{k+2}) \neq (z_{k+1}, z_{k+2})$ then $x_{k+2} \notin [l_{k+2}, u_{k+2}]$. In this case, the encoder attempts to inform the decoder of the error by transmitting a backspace. That is, it selects $(z_{k+3}, z_{k+4}) = (-1, 1)$. If these two symbols are received correctly then, according to (7-9), the decoder will update the estimate by first backing up a step. There could, however, been an error in sending the backspace as well. The encoder will keep sending backspaces until $\hat{k_0} - N = k_0$. At this time, say k', observe that $x_{k'} \in [l_{k'}, u_{k'}]$. Evidently, the encoder will mark $k_0 = k'$ and will attempt to send real estimates once again.

We abstract the decoder state as (i, j) to mean that the decoder has received i correct estimates followed by j erroneous estimates. Careful examination of (7) reveals that the state (i, j) has the Markov property and may be described by the Markov chain illustrated in Figure 1. The transition probabilities are obtained from the encoding of $\{-1, 1, B\}$ together with the error probability e of the BSC, see Figure 2. The m-th moment of $|x_k - \hat{x}_k|$ is given by the expression

$$\sum_{i+j \leq k} \frac{|a^2|^{mk}}{2^{m(i-j)}} p_k(i, j) \tag{11}$$

where at time k, $p_k(i, j)$ is the probability of being in state (i, j) and the summation is over all $i + j \leq k$ since at most i correct and j incorrect symbols could have been received by the decoder. We have evaluated this expression for $m = 2$ (the variance) and several values of $|a|$. The results are plotted in Figure 3. We cannot determine the analytic bounds imposed on the system by the BSC via simulation results. However, we do see that the backspace protocol together with a 2-bit encoding provides values of $|a| > 1$ for which the variance converges. There is a respectable performance degradation from the bounds for the BEC to the values of $|a|$ computed for the BSC for which the variance is bounded. Note that the large gap may be attributed to the overhead of using twice as many bits to encode one extra symbol (namely the back-space symbol). This overhead would significantly be reduced if the original alphabet consisted of say M symbols for larger values of M. That is, the ratio

$$\frac{2^{\lceil \log_2 M+1 \rceil}}{2^{\lceil \log_2 M \rceil}}$$

would be much smaller than 2.

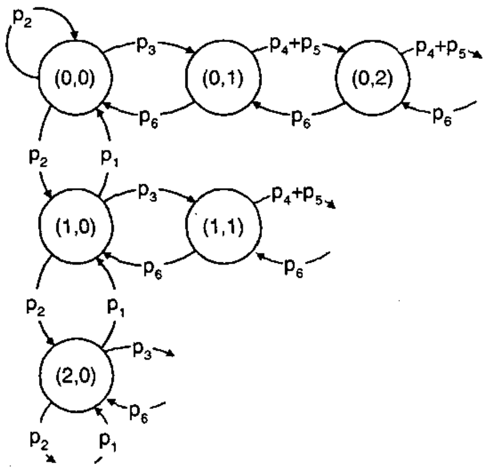

Figure 1: Markov chain describing the abstract state (i, j) of the decoder.

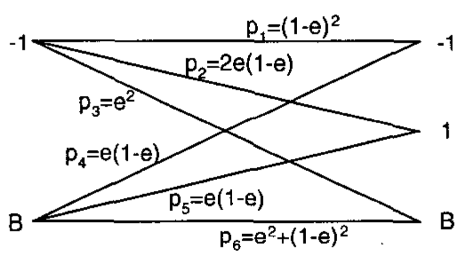

Figure 2: Error probabilities for BSC.

2.3 Estimation for unknown plant

Consider again the uncontrolled system (1) and suppose that a is unknown to the decoder. What are the restrictions imposed by the channel in this case? It should be clear that $|a| < 2$ is still necessary to keep the estimation error bounded when x_0 is not known to the decoder.

Theorem 4 Suppose x_0 and a are unknown to the decoder but bounded by $|x_0| \leq \Delta$ and $|a| < \sqrt{2}$. Suppose that $w_k = 0$ for all $k \geq 0$. Then there is a bounded-error encoder-decoder pair.

Proof Take the binary representations $x_0 = \Delta \sum_{i=0}^{\infty} \alpha_i 2^{-i}$ and $a = \sqrt{2} \sum_{i=0}^{\infty} \beta_i 2^{-i}$, with α_i, β_i as 0 or 1. For k even, the encoder selects the $\frac{k}{2}$th bit in the binary expansion $z_{k+1} = \alpha_{\frac{k}{2}}$ and for k odd, $z_{k+1} = \beta_{\frac{k-1}{2}}$. Upon receiving z_{k+1} the decoder estimates

$$\hat{x}_k = \hat{a}_k^k \hat{x}_{0k}$$

where $\hat{x}_{0k} = \Delta \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \alpha_i 2^{-i}$ and $\hat{a}_k = \sqrt{2} \sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \beta_i 2^{-i}$. Thus,

$$\begin{aligned}
|x_k - \hat{x}_k| &= |a^k x_0 - \hat{a}_k^k \hat{x}_{0k}| \\
&= |(a^k - \hat{a}_k^k) x_0 - \hat{a}_k^k (\hat{x}_{0k} - x_0)| \\
&\leq |(a - \hat{a}_k) k a^{k-1}||x_0| + |\hat{a}_k^k||x_{0k} - x_0| \\
&\leq k\Delta 2^{-\lfloor \frac{k-1}{2} \rfloor}|a^{k-1}| + |\hat{a}_k^k| 2^{-\lfloor \frac{k}{2} \rfloor}.
\end{aligned}$$

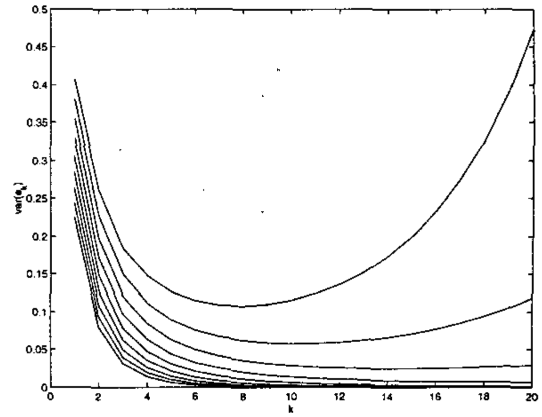

Figure 3: Variance of the error $e_k = |x_k - \hat{x}_k|$ for different values of $1 \leq |a| \leq 1.16$.

which remains bounded when $|a| < \sqrt{2}$. □

The encoder-decoder pair of Theorem 4 assumes that a and x_0 are precisely known by the encoder. This scheme has two disadvantages: it is not robust against encoder error, and does not extend to the case with unknown disturbance. In the previous cases of known a the encoder-decoder pair was presented in feedback form where at time k the encoder was selecting values for z_{k+1} based on a kind of *innovation.*

There is a gap between the necessary ($|a| < 2$) and sufficient conditions ($|a| < \sqrt{2}$) for a bounded-error encoder-decoder pair to exist. Intuitively, if we insist on getting across accurate estimates for both $\hat{x}_0$ and $\hat{a}$ it seems that the sufficient condition is also necessary. However, we know from the adaptive control literature that there are *self-tuning* control laws that do not require complete knowledge of every system parameter. Consider the system $x_{k+1} = ax_k + bu_k$ and let $\hat{\theta}_k = (\hat{a}_k, \hat{b}_k)^T$ denote the estimate of the system parameters. Then, if the estimate is given by the LSE

$$\hat{\theta}_k = \left[\sum_{i=0}^{k-1} \phi_i \phi_i^T \right]^{-1} \sum_{i=0}^{k-1} \phi_i x_{i+1}$$

where $\phi_i = (x_i, u_i)^T$, the certainty equivalent control

$$u_k = -\frac{\hat{a}_k}{\hat{b}_k} x_k,$$

is both self-tuning ($\frac{\hat{a}_k}{\hat{b}_k} \to \frac{a}{b}$) and self-optimizing for the cost-criterion given by $\lim \frac{1}{N} \sum_{i=0}^{N-1} x_i^2$. However, it is not necessarily the case that $\hat{\theta}_k \to (a, b)^T$. This suggests the following conjecture.

Conjecture Suppose x_0 and a are unknown to the decoder. Then $|a| < \sqrt{2}$ is a sufficient and necessary condition for there to be a bounded-error encoder-decoder pair that is independent of the control. However, there exist bounded-error control dependent encoder-decoder pairs for which $\sqrt{2} < |a| < 2$.

3 Continuous-time systems

Section 1 summarizes a collection of results for discrete-time systems. These results are applicable to the continuous-time system,

$$dx(t) = \alpha x(t)dt + dw(t), \ x(0) = 0, \qquad (12)$$

by the usual *discretization* procedure and a discrete controller. Such a controller has several shortcomings when there are communication constraints (suppose the sampling period is T): (1) Requiring the communication link to get one bit across every T seconds may be too restrictive; (2) If the channel is not digital, then the analysis is in general very difficult; (3) Networks are not characterized by digital channels.

Walsh, Branicky and Nilsson [5, 6] studied networked control systems by abstracting the communication links as (variable) delay elements. Their approach is also based on discretization. Their results show that controller designs that are robust against a range of unknown delays leads to very conservative choices. Much better performance can be achieved if delays are known (but random). However, unlike the approach of Tatikonda and Sahai, they do not include a formal characterization of the network links.

In the spirit of Astrom we propose an event-based control structure that is again constrained by a communication link. We take the communication link to be a a single-server FIFO queue, see figure 4. The queue is characterized by its service distribution. The exponential (or memoryless) server is denoted as G/M/1 where G denotes the (general) i.i.d. interarrival times τ_n and M denotes the i.i.d. service times, exponentially distributed with mean μ^{-1}.

We are interested in a complete information-theoretic characterization of the capacity constraints for networked control systems. As Sahai shows Shannon capacity is not the right figure of merit for closed-loop discrete-time systems, we also show that the Shannon capacity of a queue does not characterize stability. For a G/M/1 server and $\alpha = 0$ we characterize the stability constraints in terms of the service rate μ. Furthermore, since queueing networks are good mathematical models of many communication networks, our approach offers an alternate formulation to [5] for networked control systems.

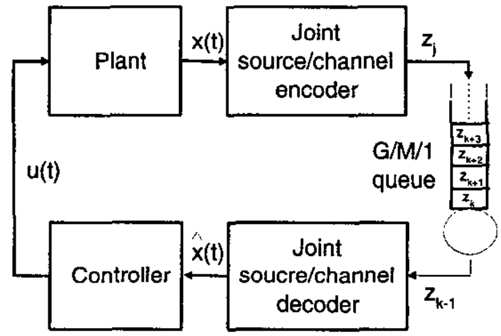

Figure 4: The continuous-time event-based control problem over a queue.

Consider system (12). Take $w(t)$ to be Brownian motion, $\alpha = 0$, and for simplicity take $x(0) = 0$. Fix a threshold $\epsilon > 0$, and recursively define ($t_0 = 0$)

$$t_k = \inf\{\, t > t_{k-1} \ : \ |x(t) - x(t_{k-1})| \geq \epsilon \,\},$$

for $k \geq 1$. At time t_k the encoder sends bit $z_k = \pm 1$ accordingly as $x(t_k) - x(t_{k-1}) = \pm\epsilon$. Because of the service distribution the queue may introduce delays. Suppose at time t the decoder has received the observations $z^k = (z_1, \ldots, z_k)$, $t_1 < \cdots < t_k < t$, the decoder's estimate $\hat{x}(t)$ is

$$\hat{x}(t) = \sum_{i=1}^{k} z_k \epsilon.$$

Evidently, if at time t the encoder has generated n_1 bits and the decoder has received $n_2 \leq n_1$ bits,

$$|x(t) - \hat{x}(t)| \leq (n_1 - n_2 + 1)\epsilon.$$

Theorem 5 If the communication channel is a G/M/1 queue with service rate μ such that

$$\frac{1}{\mu} < \epsilon^2$$

then

$$\sup_{k>0} E|x(t) - \hat{x}(t)| < \infty.$$

Proof Denote the interarrival times by $\tau_k = t_k - t_{k-1}$ and let $H(t) = P\{\tau_k \leq \tau\}$ for $\tau \geq 0$. Consider the Laplace transform

$$H^*(s) = E(e^{-s\tau_k}) = \int_0^\infty e^{-s\tau} dH(\tau).$$

We know from the theory of queueing networks [7] that the G/M/1 queue is equivalent to a birth-death process whose balance equations are satisfied by $\pi(i) = (1-\nu)\nu^i$, $i \geq 0$ where ν is the fixed-point solution of $\nu = H^*(\mu - \mu\nu)$. Here $\pi(n)$ denotes the invariant probability that the queue length is n. We also know that the birth-death process has at most one invariant distribution and a sufficient condition for one to exist is $\frac{1}{\mu} < E(\tau_k)$.

Now, when $\alpha = 0$, $x(t)$ is Brownian motion and $t - x(t)^2$ is a martingale so that

$$E(t_k - t_{k-1}) = E((x(t_k) - x(t_{k-1}))^2) = \epsilon^2.$$

The interarrival times are i.i.d. so that the queue is indeed G/M/1 and if $\frac{1}{\mu} < \epsilon^2$ then an invariant distribution exists so that the expected queue length is finite:

$$\sup_{k>0} E|x(t) - \hat{x}(t)| < \epsilon \left(\frac{\nu}{\nu + 1} + 1 \right) < \infty.$$

□

Thus we see that when $\alpha = 0$, the input rate to the queue is given by ϵ^2 and for any service rate μ we can always pick $\frac{1}{\mu} < \epsilon^2$. As an example, take $\mu = 1$. From [9] we know that $H^*(s)$ for Brownian motion is given by

$$H^*(s) = \frac{1}{\cosh(\epsilon\sqrt{2s})}.$$

Figure 5 illustrates the expected estimation error and its variance for different values of ϵ. The trade-off is that a larger ϵ causes smaller queue build-ups but a larger variance.

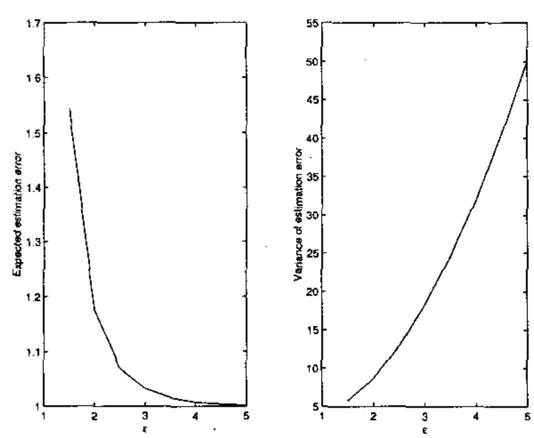

Figure 5: Estimation error and variance for $\mu = 1$.

To see the information-theoretic implications of this result, consider the Shannon capacity of a G/G/1 queue computed by Anantharam [8]. If the G/G/1 queue has *deterministic* service times then it has infinite capacity. Denote this service time by μ. It is clear, however, that if $\epsilon^2 < \frac{1}{\mu}$ then the queue length will tend to infinity so that the estimation error grows unbounded. This shows that for the estimation scheme above, the Shannon capacity of the channel does not produce the right figure of merit.

The case $\alpha \neq 0$ is more complicated and is the subject of current research.

4 Conclusion

We investigated variations of the simple discrete-time linear system (1) for which the communication between the sensor and the controller is constrained by a binary communication channel. For the digital (or error-free) binary channel we derived the bound $|a| < 2$ which is valid for both the state-estimation error to converge and the stabilizability of the system.

We further investigated the bounds on $|a|$ for the binary erasure (BEC) and binary symmetric channels (BSC). For the BEC we reproduced the analytic results of Tatikonda and Sahai which are valid for the state-estimation error to be bounded. We compared the results to the Shannon capacities of the binary channels. For the BSC, we proposed a *backspace* protocol for the encoder-decoder and demonstrated various values of $|a|$ for which the variance of the state-estimation error remains bounded.

We concluded our investigation of discrete-time linear systems by discussing the case when the state transition matrix a is unknown to the controller (or decoder).

For the discrete-time case, the bounds on $|a|$ for the controlled system (3) is an open problem for both the BEC and the BSC. The case of unknown a also requires further research.

For the continuous-time system (12) we considered a binary queue as the communication link between the sensor and the controller. For the case $a = 0$, we characterized the bound on the system using the service rate μ of the queue. Once again, we compared the result to the Shannon capacity of the queue. The case $a > 0$ is an open problem.

5 Acknowledgment

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