On the existence of competitive equilibria in bandwidth markets*

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Abstract

The recent collapse of bandwidth markets motivates our study. We describe the economic framework for such a market. Examples demonstrate the problems in finite markets in which bandwidth trades in indivisible units. Competitive equilibria exist under the continuum model of perfect competition. We explore the countably inifinite economy as a model of perfect competition for such markets. Finally, we show existence of various enforceable and non-enforceable approximate competitive equilibria.

1 Introduction

For 18 months beginning summer, 2000, it seemed that organized bandwidth markets would grow to dominate bandwidth trading. Economic incentives favored such markets: New suppliers (Global Trading, Qwest) competed with established carriers (AT&T, WorldCom); and many customers (ISPs, VoIP companies, enterprises) demanded different services (telephony minutes, IP transport, data transport with QoS guarantees). The trade press expected bandwidth markets to promote efficiency, price transparency, and price stability through the creation of derivatives.

Established carriers resisted bandwidth 'commoditization'. It upset their traditional approach of making individual business deals with their large customers. But they recognized that their oligopolistic practices would have to change. Some saw in bandwidth markets the means to reach a more objective valuation of their network assets on the basis of future earnings, rather than on past investments.

The debacle of Enron Broadband Services and other traders aborted the nascent markets. In retrospect, in addition to corporate fraud, there seem to be more fundamental causes of the collapse. There is a large 'oversupply' of bandwidth, especially as demand from dotcom companies evaporated.[1] Technical difficulties (insufficient peering points, lack of standardized products except for SONET, inability to guarantee end-to-end QoS) also hampered the acceptance of bandwidth trading [11].

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[1] "In its June, 2001 newsletter, Enron priced an OC-3 from New York City to Los Angeles, for July delivery, at \$29,587 per month. By mid-July, the price was down to \$16,437 a month for September delivery, and an amazing \$1,879 per month for delivery in January 2002." [11]. These prices may be misleading, however, in that the volume of trade is not known.

Our aim is to initiate a study of bandwidth markets within mathematical models that emphasize features which distinguish them from other markets. One feature is that bandwidth is traded in indivisible units: One can buy or sell multiple trunks on a link, each with a fixed bandwidth, say OC-3, but not a fraction of a trunk. Another feature is its combinatorial nature: A buyer must purchase bandwidth in several links to create an end-to-end route; a seller owning two links may wish to sell both at a price less than the sum of his offers on each link separately. (Indivisibility and combinatorial nature are also characteristic of spectrum and logistics auctions.)[2]

We formulate bandwidth markets as double auctions, which allows room for a study of agents' strategic behavior. However, our emphasis here is on the existence or non-existence of competitive equilibria, that is, allocations in which aggregate demand equals aggregate supply, and prices that support allocations. (The study of strategic behavior is more difficult and under current investigation.)

The set-up is as follows: Consider a network $G = (N, L)$ with finite sets of nodes, N, and links, L. A link's capacity is an integer number of trunks, each trunk having the same bandwidth. There are M agents, each with an initial endowment of money, and another agent 0, who owns the whole network. We could also imagine it to be a double auction with buyers and sellers, buyers specifying the bundle of links (comprising a route), the bandwidth (number of trunks) on each link, and the price they are willing to pay for the bundle, whereas the sellers specify a similar bundle and the minimum price they are willing to accept. Agents' prefences are monotonic over the bundle (they prefer larger bundles to smaller ones), and continuous in money. Buyers insist on getting the same bandwidth on all links in their bundles.

We first give an example of a finite network-finite agent economy, for which no competitive equilibrium exists. We then model a perfect competition economy as one with a continuum of agents, each with negligible influence on the final allocation and prices [8, 9]. We show that competitive equilibria exist in a continuum economy with money. (But if there is no money, there may be no competitive equilibrium.) This is accomplished using the Debreu-Gale-Nikaido lemma. We then investigate the model when there are countably infinite number of agents and goods. We finally show the existence of various non-enforceable approximate equilibria when competitive equilibria do not exist.

Literature review

Most of the related literature considers the computational aspects of combinatorial auctions, well surveyed in [12]. Closely related to auctions is the theory of classical mechanism design, a good account of which is in [1]. Papers by Kelly [2, 4, 3], Low [7], and others [10] deal with pricing of communication networks services, with an emphasis on achieving efficiency in the face of congestion externalities. However, our approach seems to be the first to consider explicitly a general equilibrium setting treating bandwidth as indivisible. Kelly's work comes close to our approach but treats bandwidth as divisible.

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[2] Other distinguishing features seem important but we are unable to usefully incorporate them in our model: QoS as embodied in SLAs; network topology, more specifically, the presence or absence of peering points; and the role of BGP. Multiple SLAs supported on the same link might be viewed as 'joint' products. Peering points determine whether links may be combined into alternative routes; they thereby affect supply elasticity since buyers want end-to-end routes. BGP plays an analogous role since the protocol determines which routes are permitted.

2 Finite Economies

We first consider a finite network economy. From general equilibrium theory [6], we know that a competitive equilibrium exists when preferences are convex, monotone and continuous (along with some boundary conditions). Kelly [2, 3] exploits this fact to show the existence of prices for bandwidth on each link that maximize a particular system objective (or social welfare) function. The prices are competitive equilibrium prices. The result holds even in the presence of network loops. But the result does not hold if link bandwidth is not divisible, as seen in the examples below.

Single trunk links

All links carry a single trunk. Buyer i wants route R_i comprising one or more links which he values at α_i. The utility is additive in money. We say a matrix is *totally unimodular* (TU) if the determinant of every square submatrix is 0, 1 or -1. A *network is TU* when the edge-route incidence matrix for the network digraph is TU. The property, of course, depends on the set of routes that buyers want, and so a non-TU network may be TU for another set of routes. In the general case, we consider routes for all source-destination pairs.

Proposition 1 (TU network). *If utilities are linear in bandwidth, and network is TU, then competitive equilibrium always exists.*

This follows because in this case LP relaxation solves the integer program. The associated Lagrange multipliers are the prices. Not every network is TU though. If the set of links formed by the routes includes a cycle, the network is not TU. (A network without cycles may also be non-TU.)

Example 1 (Non-TU networks). *Consider the cyclic network in figure 1 with the set of routes under consideration being* $\{\{e1,e2\},\{e2,e3\},\{e3,e1\}\}$*. Also consider the four node network in figure 2 with no cycles with the set of routes* $\{\{e1,e3\},\{e2,e3\},\{e1,e4,e2\}\}$*. In both cases, the incidence matrix is given by*

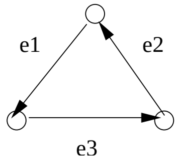

Figure 1: A cyclic network that is not TU.

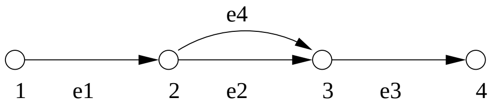

Figure 2: An acyclic network that is not TU.

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

which is not TU since it has a square submatrix with determinant 2. ∎

If the network is not TU, the existence of competitive equilibrium is not guaranteed as the following example shows.

Example 2 (Non-existence of C.E.). *(1) For the cyclic network of figures 1 with one trunk on each link, suppose the demanded routes are* $\{e1,e2\}$,$\{e2,e3\}$, $\{e3,e1\}$ *and* $\{e3\}$ *with utility 1 for first three agents for the demanded bundle (zero otherwise), and* $\alpha(< 0.5)$ *for the fourth. The network is not-TU, and there is no competitive equilibrium: The market-clearing allocation is e1 and e2 to user 1 and e3 to user 4. If prices* p_1, p_2, p_3 *were to support this allocation, they must satisfy the conditions,* $1 = p_1 + p_2 \leq \min(p_2 + p_3, p_1 + p_3)$ *and* $0.5 \geq \alpha \geq p_3$*, which is impossible.*
(2) For the acyclic network of figure 2 four users have these route-utility combinations: R_1=$\{e1,e3\}$, R_2=$\{e3,e2\}$, R_3=$\{e1,e4,e2\}$, *and* R_4=$\{e3\}$ *and utilities 1,1,1, and* $\alpha(< 0.5)$ *for the demanded bundle (zero otherwise). Because there is only one demand for e4, its equilibrium price is zero, and the example reduces to the previous one. Again, there is no competitive equilibrium.* ∎

Multiple trunk links

Example 3 (Non-existence of C.E.). *(1)* Cyclic network: *Consider the same cyclic network with 3 trunks on each link. Let the demanded routes be as before with utilities* $\sqrt{x}$ *for* x *trunks, for the first three users, and utility* $\alpha\sqrt{x}$ *for the fourth user,* $\alpha < 0.5$*. Again a competitive equilibrium does not exist. (If* x *is allowed to be divisible, an equilibrium does exist.)*
(2) Acyclic network: *Consider the same acyclic network with 3 trunks on each link. Let the demanded routes be as before with utilities* $\sqrt{x}$ *for* x *trunks, and utility* $\alpha\sqrt{x}$ *for the fourth user,* $\alpha < 0.5$*. Again a competitive equilibrium does not exist.* ∎

3 Continuum Economies

We now model a communication network with indivisible trunks as an infinite economy in which perfect competition exists, i.e., no single agent can influence the outcome. We model perfect competition by a continuum of agents such as often used in general equilibrium theory. Of course, we will to appropriately scale the quantities of goods as well.

With money:

We consider an exchange economy $\mathcal{E}$ with L indivisible (discrete) goods $(1, ..., L)$ and one divisible good 0 (money). We assume a continuum of agents $t \in X = [0, M]$, with a given non-atomic measure space $(X, \mathcal{B}(X), \mu)$. In the context of bandwidth auctions, we assume that agent 0 is the auctioneer who owns the network, and whose utilities are linear in money (i.e., zero for the links). There are M routes and agents $t \in (m, m + 1]$ demand route $m + 1$, for $m + 1 \leq M$. We assume that the preferences are monotonic, and continuous in money. This results in preferences being continuous. In particular, we can assume that preferences are quasi-linear in money. The initial endowments are denoted by ω_t, which is a $L + 1$-tuple. For agent 0, $\omega_0 = (0, C_1, ..., C_L)$, where C_l is the capacity of link l. For an agent $t(> 0)$, $\omega_t = (m_t, 0, ..., 0)$, where m_t is the money endowment of agent t. Similarly, the price vector $p = (p_0, p_1, ..., p_L)$ is a $L + 1$-tuple, where p_0 is the price of money (as usually considered in general equilibrium analysis) and p_l is the price of unit bandwidth on link l.

We begin with a few definitions: Let $p \in \Theta = \mathbb{R}_+^{L+1}$ be a price vector.
Commodity space: $\Omega = \mathbb{R}_+ \times \mathbb{Z}_+^L$.

Unit simplex: $\Delta = \{p \in \Theta : \Sigma_0^L p_l = 1\}$.
Budget set: $B_t(p) = \{z \in \Omega : p.z \leq p.\omega_t\}$.
Preference level sets: $\mathcal{P}(z) = \{z' \succeq_t z\}$.
Individual demand correspondence: $\psi_t(p) = \{z \in B_t(p) : z \succeq_t z', \forall z' \in B_t(p)\}$.
Aggregate excess demand correspondence:

$$\Phi(p) = \int_X \psi_t(p) d\mu - \int_X \omega_t d\mu.$$

We denote the first term by $\Psi(p)$ and the second term above by $\bar{\omega}$, the total endowment of all agents.

Definition 1 (Competitive Equilibrium). *A pair (x^*, p^*) with $p^* \in \Delta$ and $x^* \in \Omega$ is a competitive equilibrium if $x_t^* \in \psi_t(p^*)$ and $0 \in \Phi(p^*)$.*

Under mild standard conditions, the following assumptions can be proved for the economy under consideration but we state them as assumptions to stress the dependence of the existence proof on these assumptions [1].

Assumptions

1. $\bar{\omega} \gg 0$ (component-wise positive), and $\omega_t > 0, \forall t$ (component-wise non-negative with some component positive).
2. $\Phi(p)$ is homogeneous in p.
3. Boundary condition: Suppose $p^\nu \to p^*$, and $p_l^* = 0$ for some l. Then, $z_l^\nu \to \infty$, $\forall z^\nu \in \Phi(p^\nu)$.
4. Walras' Law holds: $p.z = 0, \forall z \in \Phi(p), \forall p \in \Delta^0$, the relative interior of Δ.

Theorem 1 (Existence). *In the continuum exchange economy $\mathcal{E}$, a competitive equilibrium exists.*

Proof. Consider any non-empty, closed convex subset S of Δ.

Claim 1. *Φ is non-empty and convex-valued on S.*

From assumption 1, Φ is non-empty. Fix any $p \in S$. By Lyapunov's theorem [13] with μ a non-atomic measure on X, and $\psi_t(p)$ a correspondence for each p, $\int_X \psi_t(p) d\mu(t)$ is convex. Hence, Φ is convex.

Claim 2. *Φ is compact-valued, and hence bounded on S.*

Note that S is compact and for each $p \in S$, $p \gg 0$. Write

$$\psi_t(p) = \bigcap_{z \in B_t(p)} [B_t(p) \cap \mathcal{P}(z)]. \tag{2}$$

Then, $\mathcal{P}(z)$ is closed by continuity of preferences. $B_t(p)$ is closed and bounded for $p \gg 0$. Thus, their intersection is closed. And so is the outer intersection. It is bounded as well. Thus, $\psi_t(p)$ is compact for each $p \gg 0$.

Claim 3. $p \cdot z \leq 0, \forall p \in \Delta^0$, $z \in \Phi(p)$.

Fix $p \in \Delta^0$. By definition,

$$p \cdot z \leq p \cdot \omega_t, \forall z \in \psi_t(p), \forall t \in X.$$

Or, with an abuse of notation:

$$\int_X p \cdot \psi_t(p) d\mu \leq \int_X p \cdot \omega_t d\mu,$$

$$p \cdot \Psi(p) \leq p \cdot \bar{\omega},$$

$$p \cdot \Phi(p) \leq 0,$$

Claim 4. *ψ_t is closed and upper semi-continuous (u.s.c.) in S $\forall t \in X$. Hence, Φ is closed and u.s.c. in S.*

Fix $t \in I$. To show ψ_t is closed, we have to show that for any sequences, $\{p^\nu\}, \{z^\nu\}$, $[p^\nu \rightarrow p^0, z^\nu \rightarrow z^0, z^\nu \in \psi_t(p^\nu)] \implies z^0 \in \psi_t(p^0)$. From the definition of demand correspondence, $p^\nu \cdot z^\nu \leq p^\nu \cdot \omega_t$. Taking limit as $\nu \rightarrow \infty$, we get $p^0 \cdot z^0 \leq p^0 \cdot \omega_t$, i.e. $z^0 \in B_t(p^0)$. It remains to show: $z^0 \succeq_t z, \forall z \in B_t(p^0)$.

Consider any $z \in B_t(p^0)$. Then
Case 1: $p^0 \cdot z < p^0 \cdot \omega_t$.
Then, for large enough ν, $p^\nu \cdot z < p^\nu \cdot \omega_t$. This implies that $z \in B_t(p^\nu)$. Now, $z^\nu \in \psi_t(p^\nu)$. Hence, $z^\nu \succeq_t z$. And by continuity of preferences, we get $z^0 \succeq_t z$.
Case 2: $p^0 \cdot z = p^0 \cdot \omega_t$.
Define $z'^\nu := ((1 - 1/\nu)z_0, z_1, ..., z_L) \in \Omega$, by divisibility of money. So, $p^0 \cdot z'^\nu < p^0 \cdot \omega_t$. Then, by the same argument as above: $z^0 \succeq_t z'^\nu$. And by continuity of preferences, we get $z^0 \succeq_t z$.

This implies $z^0 \in \psi_t(p)$, i.e. it is closed. Now, to show it is upper semi-continuous, we have to show by Theorem 1, page 24, [13], that for any sequence $p^\nu \rightarrow p^0$, and any $z^\nu \in \psi_t(p^\nu)$, there exists a convergent subsequence $\{z^{\nu_k}\}$ whose limit belongs to $\psi_t(p^0)$.

Now, $p^\nu \rightarrow p^0 \gg 0$. Hence, $\exists \nu_0$ s.t. $p^\nu \gg 0, \forall \nu > \nu_0$. Define

$$\pi := \inf\{p_l^\nu : \nu > \nu_0, l = 0, ..., L\}.$$

Then, $p^\nu \cdot z^\nu \leq p^\nu \cdot \omega_t$ implies for all $\nu > \nu_0$,

$$0 < z^\nu \leq \frac{p^\nu \cdot \omega_t}{\pi},$$

i.e. the sequence $\{z^\nu\}$ is bounded. Thus, by the Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{z^{\nu_k}\}$ converging to say, z^0. Since ψ_t is closed in S, $z^0 \in \psi_t(p^0)$. Thus, it is upper semi-continuous in S.

We now show that Φ is u.s.c (hence closed) as well. Let $p^\nu \rightarrow p^0$ in S. Consider $\xi^\nu \in \Psi(p^\nu) = \int_X \psi_t(p^\nu) d\mu$. Then, $\exists z_t^\nu$ s.t. $\xi^\nu = \int_X z_t^\nu d\mu$. Now, ψ_t is compact-valued and u.s.c. in S. Thus, by Theorem 1, page 24, [13], the sequence $\{z_t^\nu\}$ has a convergent subsequence $\{z_t^{\nu_k}\}$ s.t. $z_t^{\nu_k} \rightarrow z_t^0 \in \psi_t(p^0)$. Define $\xi^0 := \int_X z_t^0 d\mu$. Thus,

$$\xi^0 \in \int_I \psi_t(p^0) d\mu = \Psi(p^0).$$

As argued before, Ψ is compact-valued. Hence, by reapplication of the same theorem, it is u.s.c. in S. And so is Φ.

Then, using the Debreu-Gale-Nikaido theorem [13, 5] stated in the appendix, we get the following proposition.

Proposition 2. *For any non-empty, closed convex subset* S *of* Δ^0, $\exists p^0 \in S, z^0 \in \Phi(p^0)$ *s.t.* $p \cdot z^0 \leq 0, \forall p \in S$.

Consider a increasing sequence of sets $S^\nu \uparrow \Delta$. Let p^ν, z^ν be those given by the above proposition. Then, $p^\nu \in S^\nu \subset \Delta$, which is compact. Thus, $\exists$ a convergent subsequence $p^{\nu_k} \rightarrow p^* \in \Delta$.

Without loss of generality, consider this subsequence as the sequence. Consider any $z^\nu \in \Phi(p^\nu)$. We have the following lower bound on the sequence

$$z^\nu \geq -\bar{\omega}, \forall \nu. \tag{3}$$

To get an upper bound, take any $\tilde{p} \gg 0 \in S^\nu$. It exists because $S^\nu \uparrow \Delta$. Then, using the above proposition, we get

$$\tilde{p} \cdot z^\nu \leq 0, \tag{4}$$

for large enough ν. Equations (3) and (4) imply $\{z^\nu\}(\subset \Omega)$ is bounded. Thus, $\exists$ a convergent subsequence with limit say, z^*.

By assumption 1, $\bar{\omega} \gg 0$. Also, $p^* \in \Delta$. Hence, $p^* \cdot \bar{\omega} > 0$. Further, $p^* \gg 0$ since if $p^*_l = 0$ for some l, $z_l^{\nu_k} \rightarrow \infty$, by the boundary condition, which then contradicts the boundedness of the subsequence above. Further, since Φ is closed: $z^* \in \Phi(p^*)$.

This establishes the following lemma.

Lemma 1. $\exists p^* \gg 0 \in \Delta, z^* \in \Phi(p^*)$ *s.t.* $p^\nu \rightarrow p^*, z^\nu \rightarrow z^*$, *and* $p \cdot z^* \leq 0, \forall p \in \Delta$.

We are now ready to prove the theorem: Walras' law implies $p \cdot z = 0$, $\forall z \in \Phi(p)$, and $\forall p \in \Delta^0$. This implies $p^* \cdot z^* = 0$. From lemma above, $p^* \cdot z^* \leq 0$, and $p^* \gg 0$. This yields $z^* = 0$.

■

We can set the price of money $p_0 = 1$, and we get the other prices in units of money. Since we only require that preference be monotonic (and not strictly monotonic), it follows that:

Corollary 1. *If demands are for source-destination pairs instead of for routes, the theorem 1 still holds. And in that case, prices for various alternative routes for a given source-destination pair are same.*

Without money:

Example 4. *Consider the networks of figure 1, with demands as discussed before in example 4. Now instead of one user of each type demanding a particular route, we have a continuum of users. Let* $X = [0, 1]$ *and let all users in* $[0, \frac{1}{R}]$, *where* R *is the total number of routes, demand the same route and have identical preferences. We make the same assumption for the the other* R *disjoint intervals of length* $\frac{1}{R}$ *of the unit interval. This then reduces the continuum case to the same as example 4, and hence a competitive equilibrium does not exist.*

4 Countable Economies

We now consider an economic system where there are a countably infinite number of agents and goods. This is usually achieved by considering a sequence of finite economies. One particular way is by replication of a finite economy, i.e. if economy $\mathcal{E}$ has M agents, and N goods, then economy $\mathcal{E}^k$ has kM agents and kN goods. It is quite straightforward then to extend example 4 to show that without money, competitive equilibrium may not exist in the limit economy.

With money, of course, things are more positive. We first define the average of a finite set C

$$\bar{C}_n := \frac{1}{n}\Sigma_1^n C := \{x : \exists x_1, ..., x_n \in C, \text{ with } x = \frac{1}{n}\Sigma_1^n x_i\}$$

The limiting average of a finite set is then defined as

$$\bar{C} := \limsup_{n\to\infty} \frac{1}{n}\Sigma_1^n C.$$

Similarly, we can define the lim inf of the running average. Note that the limit may not exist. But we have the following result:

Lemma 2. *Let C be a finite set in $\mathbb{R}^m$. Then, $\bar{C}$ is convex and = conv (C).*

Proof: The proof is using a probabilistic argument. Consider any $x, y \in \bar{C}$. Then, there exist sequences whose subsequences $\langle x_n\rangle, \langle y_n\rangle$ converge to x and y resp. We want to show that for any $\lambda \in (0, 1)$, $z := x\lambda + y(1 - \lambda) \in \bar{C}$.

Construct a random sequence by choosing the nth element of the sequence z_n w.p. λ from sequence $\langle x_n\rangle$, and from sequence $\langle y_n\rangle$ w.p $(1-\lambda)$. Then, the random sequence is well-defined. Moreover, by Glivenko-Cantelli lemma [**?**], the proportion of terms from the two sequences for large enough n, is $(\lambda, 1 - \lambda)$. Thus, the running average of the random sequence converges to z. This implies the desired result. ■

A similar result holds for the lim inf of the running average and is also equal to convex hull of C, but the limit of the running average doesnot exist. The above result can be easily generalized to running average of a sequence of sets $\langle C_n\rangle$ as long as the sets are subsets of some finite set.

Now, with strict monotonicity of preferences in money, we get that the individual demand correspondences for any $p \in \Delta^0$ are finite. And with finitely many types, consider a replication of the economy, with n times replication and C being the aggregate excess demand correspondence in the unreplicated economy. It is finite for any price vector in the relative interior of the unit price simplex. The above result then is, as of now, the best that we can get for convexifaction of a countable economy with indivisible goods. However, it is not enough for an existence result.

5 Approximation results

The continuum economy is a convenient model but still a mathematical fiction. We show that it can be approximated by a large (but finite) economy. Note that in the proof of Theorem 1, we have used convexity of the aggregate excess demand correspondence to apply the DGN Theorem. Thus, if we replace $\Phi(p)$ by conv $\Phi(p)$, the following result holds.

Theorem 2. *Competitive equilibrium may not exist but $\exists$ a $p^* \gg 0$, in Δ^0 s.t. $0 \in$ conv $\Phi(p^*)$.*

We can obtain several approximation results using the Shapley-Folkman and Starr theorems:

Theorem 3 (Non-enforceable, approximate equilibria). *(i) If the number of agents (m) is greater than the number of goods (n), then at prices p^*, for which $0 \in$ conv $(\Phi(p^*))$, $\exists x^i \in$ conv $\phi_i(p^*)$ s.t.*

$$\Sigma_i x^i = 0 \textit{ and } \#\{i | x^i \notin \phi_i(p)\}/m \leq n/m \to 0.$$

(ii) At prices p^* for which $0 \in$ conv $(\Sigma_i \phi_i(p^*))$, then $\exists x^i \in \phi_i(p^*)$ s.t.

$$\|\Sigma_i x^i\|^2/m \leq R/m \to 0,$$

where $R = min\{m, n\} \cdot$ greatest $rad^2 \phi_i(p^*)$.

The first is straight forward application of the Shapley-Folkman theorem [6, 13, 5] noting the compactness of the individual demand correspondences from Claim 2. It says that there exists an allocation and prices such that the number of agents who are not happy with their allocation at those prices is bounded by the number of goods. Thus, as the number of agents, increases (as in replication), the proportion of unhappy agents becomes arbitrarily small. The second is an application of the Starr theorem: It says that at prices p^*, the aggregate excess demand per agent becomes arbitrarily small as the number of agents becomes arbitrarily large.

When a set of market-clearing prices does not exist, it is useful to know whether prices under which demand can be made arbitrarily less than supply exist, while affecting agents' surplus only by a small amount. We now show that this is the case provided that the bandwidth carried by each trunk is small enough, or equivalently the number of trunks per link is large enough. The market model we use to demonstrate this is based on [2] so we follow the notation on that paper.

Assume there exists a set J of links. For each link $j \in J$, let $C_j \in \mathbb{Z}_+$ denote the number of available trunks for this link. Let R denote the set of possible routes, i.e., a set of subsets of J. A collection of routes $s \subset R$ connecting a source with a destination is associated with a user which wishes to inject traffic through the routes in that collection. His utility $U_s(x_s)$ is assumed to be an increasing, strictly concave function over the nonnegative reals. The set of all users is denoted by S. The relation of R in terms of the link set J is expressed by a 0-1 matrix $A = (A_{jr}; j \in J, r \in R)$, where A_{jr} is 1 or 0 if $j \in r$ or not respectively. Likewise we define $H = (H_{sr}; s \in S, r \in R)$, where H_{sr} is 1 or 0 if $r \in s$ or not respectively.

In order to study the loss in efficiency as a function of the amount of bandwidth per trunk, we consider a sequence of "discrete" networks indexed by N. For the N-th network, the capacity of link $j \in J$ in terms of trunks is $C_j^N = NC_j$. Each user is allowed to pick only integral multiples of trunks along his route, thus his utility is a function $U_s^N : \mathbb{Z}_+ \to \mathbb{R}$, with $U_s^N(n) = U_s(n/N)$ for each $s \in S$. Say that each trunk at link j, costs p_j for each $j \in J$, then the *cost per trunk* over the path r is $\text{Cost}(r; p) = \sum_{j \in r} p_j$. Similarly, for $s \in S$ the lowest cost route costs $\text{Cost}(s; p) = \bigwedge_{r \in s} \text{Cost}(r; p)$.

Theorem 4. *For each $\epsilon > 0$, there exist integer $N_0 > 0$ such that for every integer N larger than N_0, there exists $(n^N, m^N, p^N) = ((n_s^N)_{s \in S}, (m_r^N)_{r \in R}, (p_j^N)_{j \in J})$ where*

1. *n_s^N maximizes $U_s^N(n_s) - n_s\text{Cost}(s; p^N)$ over $\mathbb{Z}_+$ for every $s \in S$,*
2. *$Hm^N = n^N$, $Am^N \leq C^N$, $m_r^N \in \mathbb{Z}_+$ and*
3. *$U_s^N(n_s^N) + \epsilon \geq U_s(x_s^*)$, for every $s \in S$, where $x^* = (x_s^*; s \in S)$, $y^* = (y_r^*; r \in R)$ maximize $\sum_{s \in S} U_s(x_s)$ over $x \in \mathbb{R}_+^{|S|}, y \in \mathbb{R}_+^{|R|}$ under the constraints $Hy = x$, $Ay \leq C$.*

Proof. Fix $\epsilon > 0$. Without loss of generality assume $x_s^* > 0$ for every $s \in S$, and let $p^0 = (p_j^0; j \in J)$ be the equilibrium prices as they are guaranteed to exist for the divisible case by [2]. Now if the users were presented the inflated prices $p = \alpha p^0$ instead, where $\alpha > 1$, each user s will choose x_s so that it maximizes his surplus $U_s(x_s) - x_s\text{Cost}(s; p) = U_s(x_s) - x_s\alpha\text{Cost}(s; p^0)$ over $x_s \geq 0$. So by strict concavity and monotonicity of utilities, he will choose $x_s^\alpha < x_s^*$. Now if we define $y_r^\alpha = y_r^* x_s^\alpha / x_s^*$ for each $r \in s$, then we get $\sum_{r \in R} H_{sr} y_r^\alpha = x_s^\alpha$ for every s.

Also, since all entries of A are nonnegative and there are no all-zero rows, $\sum_{r\in R} A_{jr} y_j^\alpha < C_j$ for every j. By continuity and strict monotonicity of utilities, we can choose $\alpha > 1$ so that $\sum_{s\in S} U_s(x_s^\alpha) + 2\epsilon \ge \sum_{s\in S} U_s(x_s^*)$, and $x_s^\alpha > 0$ for all s.

Now for every $N > 0$, define $p_j^N = p_j/N$ for each $j \in J$, and $n_s^N = \max\{\arg\max_{n\in\mathbb{Z}_+} U_s^N(n) - n\text{Cost}(s;p)\}$ for each $s \in S$. By strict concavity we have $|x_s^\alpha - n_s^N/N| \le 1/N$. Thus it is clear that as N tends to ∞, part 3 holds, and part 1 holds for every N.

It only remains to show part 2. Since $x_s^\alpha > 0$ for all s, there exists $r \in s$ for which $y_s^\alpha > 0$ and denote it by $r(s)$. Now define $m_{r(s)}^N = n_s^N - \sum_{r\in s\setminus\{r(s)\}} \lfloor N y_r^\alpha \rfloor$ and for each $r \neq r(s)$, $m_r^N = \lfloor N y_r^\alpha \rfloor$. Note that $m_{r(s)}^N/N \to y_r^\alpha(s)$ as $N \to \infty$, so for large enough N, $m_{r(s)}^N > 0$. Also, we have $m_r^N \to y_s^\alpha \ge 0$ as $N \to \infty$ for $r \neq r(s)$. Thus, $Hm^N = n^N$, and $m^N \ge 0$ for large enough N. Furthermore, $Am^n \le C^N$ for large enough N, since $Ay_s^\alpha < C$. ■

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